

On the diameter of hyperbolic random graphs

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Abstract Large real-world networks are typically scale-free. Recent research has shown that such graphs are described best in a geometric space. More precisely, the internet can be mapped to a hyperbolic space such that geometric greedy routing performs close to optimal (Boguná, Papadopoulos, and Krioukov. *Nature Communications*, 1:62, 2010). This observation pushed the interest in hyperbolic networks as a natural model for scale-free networks. Hyperbolic random graphs follow a power-law degree distribution with controllable exponent β and show high clustering (Gugelmann, Panagiotou, and Peter. *ICALP*, pp. 573–585, 2012).

For understanding the structure of the resulting graphs and for analyzing the behavior of network algorithms, the next question is bounding the size of the diameter. The only known explicit bound is $\mathcal{O}((\log n)^{32/((3-\beta)(5-\beta))+1})$ (Kiwi and Mitsche. *ANALCO*, pp. 26–39, 2015). We present two much simpler proofs for an improved upper bound of $\mathcal{O}((\log n)^{2/(3-\beta)})$ and a lower bound of $\Omega(\log n)$.

1 Introduction

Large real-world networks are almost always sparse and non-regular. Their degree distribution typically follows a *power law*, which is synonymously used for being *scale-free*. Since the 1960’s, large networks have been studied in detail and hundreds of models were suggested. In the past few years, a new line of research emerged, which showed that scale-free networks can be modeled more realistically when incorporating *geometry*.

Euclidean random graphs. It is not new to study graphs in a geometric space. In fact, graphs with *Euclidean geometry* have been studied intensively for more than a decade. The standard Euclidean model are random geometric graphs which result from placing n nodes independently and uniformly at random on an Euclidean space, and creating edges between pairs of nodes if and only if their distance is at most some fixed threshold r . These graphs have been studied in relation to subjects such as cluster analysis, statistical physics, hypothesis testing, and wireless sensor networks [22]. The resulting graphs are more or less regular and hence do not show a scale-free behavior with power-law degree distribution as observed in large real-world graphs.

Hyperbolic random graphs. For modeling scale-free graphs, it is natural to apply a non-Euclidean geometry with negative curvature. Krioukov et al. [19] introduced a new graph model based on *hyperbolic geometry*. Similar to euclidean random graphs, nodes are uniformly distributed in a hyperbolic space

Random Graph Model	Diameter	
Sparse Erdős-Rényi [6]	$\Theta(\log n)$ [23]	
d -dim. Euclidean [22]	$\Theta(n^{1/d})$ [15]	
Watts-Strogatz [25]	$\Theta(\log n)$ [7]	
Kleinberg [18]	$\Theta(\log n)$ [20]	
Chung-Lu [10]	$\Theta(\log n)$ [10]	} power-law graphs
Pref. Attachment [2]	$\Theta(\log \log n)$ [11]	
Hyperbolic [19]	$\mathcal{O}((\log n)^{\frac{32}{(3-\beta)(5-\beta)}+1})$ [17]	

Table 1: Known diameter bounds for various random graphs. In all cases the diameter depends on the choice of the model parameters. Here we consider a constant average degree. For scale-free networks, we also assume a power law exponent $2 < \beta < 3$.¹

and two nodes are connected if their hyperbolic distance is small. The resulting graphs have many properties observed in large real-world networks. This was impressively demonstrated by Boguná et al. [5]: They computed a maximum likelihood fit of the internet graph in the hyperbolic space and showed that greedy routing in this hyperbolic space finds nearly optimal shortest paths in the internet graph. The quality of this embedding is an indication that hyperbolic geometry naturally appears in large scale-free graphs.

Known properties. A number of properties of hyperbolic random graphs have been studied. Gugelmann et al. [16] compute exact asymptotic expressions for the expected number of vertices of degree k and prove a constant lower bound for the clustering coefficient. They confirm that the clustering is non-vanishing and that the degree sequence follows a power-law distribution with controllable exponent β . For $2 < \beta < 3$, the hyperbolic random graph has a giant component of size $\Omega(n)$ [3, 4], similar to other scale-free networks like Chung-Lu [10]. Other studied properties include the clique number [14], bootstrap percolation [9]; as well as algorithms for efficient generation of hyperbolic random graphs [24] and efficient embedding of real networks in the hyperbolic plane [21].

Diameter. The diameter, the length of the longest shortest path, is a fundamental property of a network. It also sets a worst-case lower bound on the number of steps required for all communication processes on the graph. In contrast to the average distance, it is determined by a single—atypical—long path. Due to this sensitivity to small changes, it is notoriously hard to analyze. Even subtle changes to the graph model can make an exponential difference in the diameter, as can be seen when comparing Chung-Lu (CL) random graphs [10] and Preferential Attachment (PA) graphs [2] in the considered range of the power law exponent $2 < \beta < 3$: On the one hand, we can embed a CL graph in the PA graph and they behave effectively the same [12]; on the other hand, the diameter of CL graphs is $\Theta(\log n)$ [10] while for PA graphs it is $\Theta(\log \log n)$ [11]. Table 1 provides an overview over existing results. It was open so far how the

¹ Note that the table therefore refers to a non-standard Preferential Attachment version with adjustable power law exponent $2 < \beta < 3$ (normally, $\beta = 3$).

diameter of hyperbolic random graphs compares to the aforementioned bounds for other scale-free graph models. The only known results for their diameter are $O((\log n)^{\frac{32}{(3-\beta)(5-\beta)}+1})$ by Kiwi and Mitsche [17], and a polylogarithm with no explicit constant by Bringmann, Keusch, and Lengler [8].

Our contribution. We improve upon the previous results as described by the following theorems. First, we present a much simpler proof which also shows polylogarithmic upper bound for the diameter, but with a better (i.e. smaller) exponent.²

Theorem 1. *Let $2 < \beta < 3$. The diameter of the giant component in the hyperbolic random graph $\mathcal{G}(n, \alpha, C)$ is $O((\log n)^{\frac{2}{3-\beta}})$ with probability $1 - O(n^{-3/2})$.*

The proof of Theorem 1 is presented in Section 3. For a lower bound on the diameter, we prove the following theorem.

Theorem 2. *Let $2 < \beta < 3$. Then, the diameter of the giant component in the hyperbolic random graph $\mathcal{G}(n, \alpha, C)$ is $\Omega(\log n)$ with probability $1 - n^{-\Omega(1)}$.*

We point out that although we prove all diameter bounds on the giant component, our proofs will make apparent that the giant component is in fact the component with the largest diameter in the graph.

2 Notation and Preliminaries

In this section, we briefly introduce hyperbolic random graphs. Although this paper is self-contained, we recommend to a reader who is unfamiliar with the notion of hyperbolic random graphs the more thorough investigations [16, 19].

Let \mathbb{H}_2 be the hyperbolic plane. Following [19], we use the *native* representation; in which a point $v \in \mathbb{H}_2$ is represented by polar coordinates (r_v, φ_v) ; and r_v is the hyperbolic distance of v to the origin.³

To construct a hyperbolic random graph $G(n, \alpha, C)$, consider now a circle D_n with radius $R = 2 \ln n + C$ that is centered at the origin of \mathbb{H}_2 . Inside D_n , n points are distributed independently as follows. For each point v , draw φ_v uniformly at random from $[0, 2\pi)$, and draw r_v according to the probability density function

$$\rho(r) := \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1} \approx \alpha e^{\alpha(r-R)}.$$

Next, connect two points u, v if their hyperbolic distance is at most R , i.e. if

$$d(u, v) := \cosh^{-1}(\cosh(r_u) \cosh(r_v) - \sinh(r_u) \sinh(r_v) \cos(\Delta\varphi_{u,v})) \leq R. \quad (1)$$

² The conference version of this paper [13] also contained an incorrect proof of a logarithmic upper bound on the diameter for small average degrees. In particular, Lemma 14 contained a mistake where the expected value was taken over probabilities p_i that did not add up to 1. It is an open problem to close the gap between the polylogarithmic upper and logarithmic lower bound.

³ Note that this seemingly trivial fact does not hold for conventional models (e.g. Poincaré halfplane) for the hyperbolic plane.

By $\Delta\varphi_{u,v}$ we describe the small relative angle between two nodes u, v , i.e. $\Delta\varphi_{u,v} := \cos^{-1}(\cos(\varphi_u - \varphi_v)) \leq \pi$.

This results in a graph whose degree distribution follows a power law with exponent $\beta = 2\alpha + 1$, if $\alpha \geq \frac{1}{2}$, and $\beta = 2$ otherwise [16]. Since most real-world networks have been shown to have a power law exponent $2 < \beta < 3$, we assume throughout the paper that $\frac{1}{2} < \alpha < 1$. Gugelmann et al. [16] proved that the average degree in this model is then $\delta = (1 + o(1)) \frac{2\alpha^2 e^{-C/2}}{\pi(\alpha-1/2)^2}$.

We now present a handful of Lemmas useful for analyzing the hyperbolic random graph. Most of them are taken from [16]. We begin by an upper bound for the angular distance between two connected nodes. Consider two nodes with radial coordinates r, y . Denote by $\theta_r(y)$ the maximal radial distance such that these two nodes are connected. By equation (1),

$$\theta_r(y) = \arccos\left(\frac{\cosh(y)\cosh(r) - \cosh(R)}{\sinh(y)\sinh(r)}\right). \quad (2)$$

This terse expression is closely approximated by the following Lemma.

Lemma 3 ([16]). *Let $0 \leq r \leq R$ and $y \geq R - r$. Then,*

$$\theta_r(y) = \theta_y(r) = 2e^{\frac{R-r-y}{2}}(1 \pm \Theta(e^{R-r-y})).$$

For most computations on hyperbolic random graphs, we need expressions for the probability that a sampled point falls into a certain area. To this end, Gugelmann et al. [16] define the *probability measure* of a set $S \subseteq D_n$ as

$$\mu(S) := \int_S f(y) dy,$$

where $f(r)$ is the probability mass of a point $p = (r, \varphi)$ given by $f(r) := \frac{\rho(r)}{2\pi} = \frac{\alpha \sinh(\alpha r)}{2\pi(\cosh(\alpha R) - 1)}$. We further define the *ball* with radius x around a point (r, φ) as

$$B_{r,\varphi}(x) := \{(r', \varphi') \mid d((r', \varphi'), (r, \varphi)) \leq x\}.$$

We write $B_r(x)$ for $B_{r,0}(x)$. Note that $D_n = B_0(R)$. Using these definitions, we can formulate the following Lemma.

Lemma 4 ([16, 17]). *For any $0 \leq r \leq R$ we have*

$$\mu(B_0(r)) = e^{-\alpha(R-r)}(1 + o(1)) \quad (3)$$

$$\mu(B_r(R) \cap B_0(R - m)) = \frac{2\alpha}{\pi(\alpha-1/2)} \cdot e^{-\alpha m - \frac{1}{2}(r-m)} + \mathcal{O}(e^{-\alpha r}) \quad (4)$$

Since we often argue over sequences of nodes on a path, we say that a node v is *between* two nodes u, w , if $\Delta\varphi_{u,v} + \Delta\varphi_{v,w} = \Delta\varphi_{u,w}$. Recall that $\Delta\varphi_{u,v} \leq \pi$ describes the small angle between u and v . E.g., if $u = (r_1, 0), v = (r_2, \frac{\pi}{2}), w = (r_3, \pi)$, then v lies between u and w . However, w does not lie between u and v as $\Delta\varphi_{u,v} = \pi/2$ but $\Delta\varphi_{u,w} + \Delta\varphi_{w,v} = \frac{3}{4}\pi$.

Finally, we define the area $B_I := B_0(R - \frac{\log R}{1-\alpha} - c)$ as the *inner band*, and $B_O := D_n \setminus B_I$ as the *outer band*, where $c \in \mathbb{R}$ is a large enough constant.

The Poisson Point Process. We often want to argue about the probability that an area $S \subseteq D_n$ contains one or more nodes. To this end, we usually apply the simple formula

$$\Pr[\exists v \in S] = 1 - (1 - \mu(S))^n \geq 1 - \exp(-n \cdot \mu(S)). \quad (5)$$

Unfortunately, this formula significantly complicates once the positions of some nodes are already known. This introduces conditions on $\Pr[\exists v \in S]$ which can be hard to grasp analytically. To circumvent this problem, we use a Poisson point process \mathcal{P}_n [22] which describes a different way of distributing nodes inside D_n . It is fully characterized by the following two properties:

- If two areas S, S' are disjoint, then the number of nodes that fall within S and S' are independent random variables.
- The expected number of points that fall within S is $\int_S n\mu(S)$.

One can show that these properties imply that the number of nodes inside S follows a Poisson distribution with mean $n\mu(S)$. In particular, we obtain that the number of nodes $|\mathcal{P}_n|$ inside D_n is distributed as $\text{Po}(n)$, i.e. $\mathbb{E}[|\mathcal{P}_n|] = n$, and

$$\Pr(|\mathcal{P}_n| = n) = \frac{e^{-n}n^n}{n!} = \Theta(n^{-1/2}).$$

Let the random variable $\mathcal{G}(\mathcal{P}_n, n, \alpha, C)$ denote the resulting graph when using the Poisson point process to distribute nodes inside D_n . Since it holds

$$\Pr[\mathcal{G}(\mathcal{P}_n, n, \alpha, C) = G \mid |\mathcal{P}_n| = n] = \Pr[\mathcal{G}(n, \alpha, C) = G],$$

we have that every property p with $\Pr[p(\mathcal{G}(\mathcal{P}_n, n, \alpha, C))] \leq \mathcal{O}(n^{-c})$ holds for the hyperbolic random graphs with probability $\Pr[p(\mathcal{G}(n, \alpha, C))] \leq \mathcal{O}(n^{\frac{1}{2}-c})$.

We explicitly state whenever we use the Poisson point process $\mathcal{G}(\mathcal{P}_n, n, \alpha, C)$ instead of the normal hyperbolic random graph $\mathcal{G}(n, \alpha, C)$. In particular, we can use a matching expression for equation (5): $\Pr[\exists v \in S] = 1 - \exp(-n \cdot \mu(S))$.

3 Polylogarithmic Upper Bound

As an introduction to the main proof, we first show a simple polylogarithmic upper bound on the diameter of the hyperbolic random graph. We start by investigating nodes in the inner band B_I and show that they are connected by a path of at most $\mathcal{O}(\log \log n)$ nodes. We prove this by partitioning D_n into R layers of constant thickness 1. Then, a node in layer i has radial coordinate $\in (R - i, R - i + 1]$. We denote the layer i by $L_i := B_0(R - i + 1) \setminus B_0(R - i)$.

Lemma 5. *Let $1 \leq i, j \leq R/2$, and consider two nodes $v \in L_i, w \in L_j$. Then,*

$$\frac{2}{e} e^{\frac{i+j-R}{2}} (1 - \Theta(e^{i+j-R})) \leq \theta_{r_u}(r_v) \leq 2e^{\frac{i+j-R}{2}} (1 + \Theta(e^{i+j-R})),$$

Furthermore, we have $\mu(L_j \cap B_R(v)) = \Theta(e^{-\alpha j + \frac{i+j-R}{2}})$, and, if $(i+j)/R < 1 - \varepsilon$ for some constant $\varepsilon > 0$, we have for large n

$$\frac{1}{e} e^{-\alpha j + \frac{i+j-R}{2}} \leq \mu(L_j \cap B_R(v)) \leq 4e^{-\alpha j + \frac{i+j-R}{2}}.$$

Proof. The statements follow directly from Lemmas 3 and 4 and the fact that we have $R - i < r_v \leq R - i + 1$ for a node $v \in L_i$. \square

Using Lemma 5, we can now prove that a node $v \in B_I$ has a path of length $\mathcal{O}(\log \log n)$ that leads to $B_0(R/2)$. Recall that the inner band was defined as $B_I := B_0(R - \frac{\log R}{1-\alpha} - c)$, where c is a large enough constant.

Lemma 6. *Consider a node v in layer i . With probability $1 - \mathcal{O}(n^{-3})$ it holds*

1. *if $i \in [\frac{\log R}{1-\alpha} + c, \frac{2 \log R}{1-\alpha} + c]$, then v has a neighbor in layer L_{i+1} , and*
2. *if $i \in [\frac{2 \log R}{1-\alpha} + c, R/2]$, then v has a neighbor in layer L_j for $j = \frac{\alpha}{2\alpha-1}i$.*

Proof. The probability that node $v \in L_i$ does not contain a neighbor in L_{i+1} is

$$(1 - \Theta(e^{-\alpha(i+1)+i+\frac{1-R}{2}}))^n \leq \exp(-\Theta(1) \cdot e^{\log R + c(1-\alpha)}).$$

Since $R = 2 \log n + C$ and c is a large enough constant, this proves part (1) of the claim. An analogous argument shows part (2). \square

Lemma 6 shows that there exists a path of length $\mathcal{O}(\log \log n)$ from each node $v \in B_I$ to some node $u \in B_0(R - \frac{2 \log R}{1-\alpha} - c)$. Similarly, from u there exists a path of length $\mathcal{O}(\log \log n)$ to $B_0(R/2)$ with high probability. Since we know that the nodes in $B_0(R/2)$ form a clique by the triangle inequality, we therefore obtain that all nodes in B_I form a connected component with diameter $\mathcal{O}(\log \log n)$.

Corollary 7. *Let $\frac{1}{2} < \alpha < 1$. With probability $1 - \mathcal{O}(n^{-3})$, all nodes $u, v \in B_I$ in the hyperbolic random graph are connected by a path of length $\mathcal{O}(\log \log n)$.*

3.1 Outer Band

By Corollary 7, we obtain that the diameter of the graph induced by nodes in B_I is at most $\mathcal{O}(\log \log n)$. In this section, we show that each component in B_O has a polylogarithmic diameter. Then, one can easily conclude that the overall diameter of the giant component is polylogarithmic, since all nodes in $B_0(R/2)$ belong to the giant component [4]. We begin by presenting one of the crucial Lemmas in this paper that will often be reused.

Lemma 8. *Let $u, v, w \in V$ be nodes such that v lies between u and w , and let $\{u, w\} \in E$. If $r_v \leq r_u$ and $r_v \leq r_w$, then v is connected to both u and w . If $r_v \leq r_u$ but $r_v \geq r_w$, then v is at least connected to w .*

Proof. By [4, Lemma 5.28], we know that if two nodes $(r_1, \varphi_1), (r_2, \varphi_2)$ are connected, then so are $(r'_1, \varphi_1), (r'_2, \varphi_2)$ where $r_1 \leq r'_1$ and $r'_2 \leq r_2$. Since the distance between nodes is monotone in the relative angle $\Delta\varphi$, this proves the first part of the claim. The second part can be proven by an analogous argument. \square

For convenience, we say that an edge $\{u, w\}$ *passes under* v if one of the requirements of Lemma 8 is fulfilled. Using this, we are ready to show Theorem 1. In this argument, we investigate the angular distance a path can at most traverse until it passes under a node in B_I . By Lemma 8, we then have with high probability a short path to the center $B_0(R/2)$ of the graph.

(*Proof of Theorem 1*). Partition the hyperbolic disc into n disjoint sectors of equal angle $\Theta(1/n)$. The probability that k consecutive sectors contain no node in B_I is

$$\begin{aligned} (1 - \Theta(k/n) \cdot \mu(B_0(R - \frac{\log R}{1-\alpha} - c)))^n &\leq \exp(-\Theta(1) \cdot k \cdot e^{-\alpha \log R/(1-\alpha)}) \\ &= \exp(-\Theta(1) \cdot k \cdot (\log n)^{-\frac{\alpha}{1-\alpha}}). \end{aligned}$$

Hence, we know that with probability $1 - \mathcal{O}(n^{-3})$, there are no $k := \Theta((\log n)^{\frac{1}{1-\alpha}})$ such consecutive sectors. By a Chernoff bound, the number of nodes in k such consecutive sectors is $\Theta((\log n)^{\frac{1}{1-\alpha}})$ with probability $1 - \mathcal{O}(n^{-3})$. Applying a union bound, we get that with probability $1 - \mathcal{O}(n^{-2})$, every sequence of k consecutive sectors contains at least one node in B_I and at most $\Theta(k)$ nodes in total. Consider now a node $v \in B_O$ that belongs to the giant component. Then, there must exist a path from v to some node $u \in B_I$. By Lemma 8, this path can visit at most k sectors—and therefore use at most $\Theta(k)$ nodes—before reaching u . From u , there is a path of length $\mathcal{O}(\log \log n)$ to the center $B_0(R/2)$ of the hyperbolic disc by Corollary 7. Since this holds for all nodes, and the center forms a clique, the diameter is therefore $\mathcal{O}((\log n)^{\frac{1}{1-\alpha}}) = \mathcal{O}((\log n)^{\frac{2}{3-\beta}})$. \square

From the proof it follows that every component inhabiting $\Omega((\log n)^{\frac{1}{1-\alpha}})$ sectors is connected to the center. We derive the following Corollary.

Corollary 9. *Let $2 < \beta < 3$. The second largest component of the hyperbolic random graph is of size at most $\mathcal{O}((\log n)^{\frac{2}{3-\beta}})$ with probability $1 - \mathcal{O}(n^{-3/2})$.*

These bounds improves upon the results in [17] who show an upper bound of $\mathcal{O}((\log n)^{\frac{32}{(3-\beta)(5-\beta)}+1})$ on the diameter and $\mathcal{O}((\log n)^{\frac{64}{(3-\beta)(5-\beta)}+1})$ on the second largest component. As we will see in Theorem 2, however, the lower bound on the diameter is only $\Omega(\log n)$. It is an open problem to bridge this gap.

4 Logarithmic Lower Bound

Kiwi and Mitsche [17] provide a proof for the existence of a path component of length $\Theta(\log n)$ with high probability. In this section, we show a stronger statement, namely that the *largest* component has a diameter of $\Omega(\log n)$. This proves the intuition that the component with the largest diameter is in fact the giant component, which is not obvious a priori.

Proof of Theorem 2. Let $\varepsilon := (\frac{1}{2} - \frac{1}{4\alpha})$. Observe that for $\frac{1}{2} < \alpha < 1$, we have $0 < \varepsilon < \frac{1}{4}$. Consider the model $\mathcal{G}(n, \alpha, C)$, i.e. *not* the Poisson point process. With high probability, there are no nodes in $B_0(\varepsilon R)$:

$$\begin{aligned} \Pr[B_0(\varepsilon R) = \emptyset] &= \exp(-\Theta(1) \cdot e^{R/2} \cdot e^{-(\frac{\alpha}{2} + \frac{1}{4})R}) \\ &= 1 - \Theta(1) \cdot e^{(\frac{1}{4} - \frac{\alpha}{2})R}. \\ &= 1 - n^{-\Omega(1)}. \end{aligned}$$

In the following, we condition on the fact that there are no such nodes; and switch to the Poisson point process. Consider now a node $v \in L_1$. The largest angular distance v can have to one of its neighbors is

$$\Delta\varphi \leq 2e^{-\frac{\varepsilon R}{2}}(1 \pm \mathcal{O}(e^{-\varepsilon R})) \leq \mathcal{O}(n^{-\varepsilon}). \quad (6)$$

Similarly to Theorem 1, we partition the Disc D_n into $\Theta(n)$ sectors of equal angle $\varphi := e^{-R/2} = \Theta(n)$. Then, two nodes $u, v \in L_1$ in neighboring sectors have angular distance at most $2e^{-R/2}$, and are therefore connected. On the flip side, two nodes with at least 6 sectors between them have no edge, since their angle is $6e^{-R/2} > 2e^{-R/2+1}(1 + \mathcal{O}(e^{-R}))$.

Consider now p consecutive sectors, where p is to be fixed later. For each of these p sectors, the probability that it contains exactly one node in L_1 is $\geq (e^{-R/2} \cdot ne^{-\alpha}) = e^{-\Theta(1)}$, i.e. a constant smaller than 1. The probability that this node has no further neighbors (apart from the neighbors in L_1 in the other sectors) is again $e^{-\Theta(1)}$ by Lemma 4. We name these nodes v_1, \dots, v_p .

Similarly, the probability that sector $p+1$ contains exactly one node v_{p+1} in L_3 is again $e^{-\Theta(1)}$. From here, we expose a path to the inner band B_I as follows. Assume we have a node $v \in L_i$. Assume further that all nodes v_1, \dots, v_p are to the left of v . Then, we consider the probability that v has a neighbor in layer L_j for $j = \frac{1}{2\alpha-1}i$, while we condition on the fact that none of the nodes v_1, \dots, v_p have neighbors in the upper layers as stated before. By Lemma 5 this means that in layer L_j , we have not yet uncovered an angle of at least

$$\frac{2}{e}e^{(i+\frac{i}{2\alpha-1}-R)/2} - 3e^{(\frac{i}{2\alpha-1}-R)/2} \geq \Theta(1) \cdot e^{(i+\frac{i}{2\alpha-1}-R)/2},$$

as $\frac{3e}{2}e^{-3/2} < 1$. Therefore, the probability that node v has a neighbor in layer L_j that is not connected to v_1, \dots, v_p , is at least

$$1 - \exp(-\Theta(1) \cdot n \cdot e^{-\frac{\alpha i}{2\alpha-1}} e^{(i+\frac{i}{2\alpha-1}-R)/2}) = e^{-\Theta(1)} < 1.$$

In total, the probability that v_1, \dots, v_p exist as described above; and that they are connected to B_I is thereby $e^{-\Theta(p+\log \log \log n)}$.

Furthermore, by equation (6), we know that when exposing this information we at most expose an angle of $\mathcal{O}(\frac{p}{n} + n^{-\Omega(1)} + \log \log \log n \cdot (\log n)^{1/(1-\alpha)})$ of the graph. Therefore, if $\frac{p}{n} < n^{-\Omega(1)}$, we can repeat this experiment independently $n^{\Omega(1)}$ times. The probability that all of them fail is at most

$$(1 - e^{-\Theta(p+\log \log \log n)})^{n^{\Omega(1)}} = \exp(-e^{-\Theta(p)} n^{\Omega(1)}) = \exp(-n^{\Omega(1)}),$$

if $p = \Theta(\log n)$ is chosen small enough. This proves the claim. \square

5 Conclusion

We derive a new polylogarithmic upper bound on the diameter of hyperbolic random graphs; and further prove a logarithmic lower bound. This immediately yields lower bounds for any broadcasting protocol that has to reach all nodes. Processes such as bootstrap percolation or rumor spreading therefore must run at least $\Omega(\log n)$ steps until they inform all nodes in the giant component. In particular, this result stands in contrast to the average distance of two nodes in the hyperbolic random graph, which is of order $\Theta(\log \log n)$ [1, 8]. This implies the existence of a path that is exponentially longer than the average path.

Our work focuses on power law exponents $2 < \beta < 3$, but we believe that it is possible to bound the diameter for $\beta > 3$ by $\Theta(\log n)$. For other scale-free models it was also interesting to study the phase transition at $\beta = 2$ and $\beta = 3$.

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